



## Codes pseudo-lineaires

C. Carlet

### ► To cite this version:

| C. Carlet. Codes pseudo-lineaires. RR-1334, INRIA. 1990. inria-00075226

**HAL Id: inria-00075226**

**<https://hal.inria.fr/inria-00075226>**

Submitted on 24 May 2006

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



UNITÉ DE RECHERCHE  
INRIA-ROQUENCOURT

# Rapports de Recherche

N° 1334

## *Programme 1*

*Programmation, Calcul Symbolique  
et Intelligence Artificielle*

## CODES PSEUDO-LINEAIRES

Claude CARLET

Novembre 1990



★ R R - 1 3 3 4 ★

Institut National  
de Recherche  
en Informatique  
et en Automatique

Domaine de Voluceau  
Roquencourt  
B.P. 105

78153 Le Chesnay Cedex  
France

Tel (1) 39 63 55 11



# CODES PSEUDO-LINEAIRES

## PSEUDO-LINEAR CODES

Claude Carlet<sup>1</sup>  
Université d'Amiens, France

### ABSTRACT

We introduce a new class of binary codes, called pseudo-linear, whose weight enumerators are changed by the Mac Williams transform into the weight enumerators of some codes, called their pseudo-dual codes.

We deduce a new proof - and an explanation - of the formal duality between the Kerdock codes and the generalized Preparata codes of same lengths, and between the Delsarte-Goethals codes  $\mathcal{DG}(m, (m-2)/2)$  and the Goethals codes  $\mathcal{T}(m)$ .

### RESUME

Nous introduisons une nouvelle classe de codes binaires : celle des codes pseudo-linéaires. Le transformé de Mac Williams du polynôme énumérateur des poids d'un code pseudo-linéaire est le polynôme des poids d'un code, son pseudo-dual. Nous en déduisons une nouvelle preuve - et une explication - de la dualité formelle des codes de Kerdock et de Preparata généralisés de même longueur d'une part, et des codes  $\mathcal{DG}(m, (m-2)/2)$  de Delsarte-Goethals et  $\mathcal{T}(m)$  de Goethals d'autre part.

---

<sup>1</sup> Chercheur Extérieur - Programme 1  
INRIA Bat 10, Domaine de Voluceau BP 105, 78153 Le Chesnay, France

## INTRODUCTION

Let  $n$  be a positive integer. We denote by  $\langle , \rangle$  the usual dot product on  $F^n$ , where  $F = \{0, 1\}$ .

A binary code of length  $n$  is a subset of  $F^n$ . Its weight enumerator :

$$W_C(X, Y) = \sum_{a \in C} X^{n-w(a)} Y^{w(a)}, \text{ where } w(a) \text{ denotes the Hamming weight of the word } a,$$

satisfies the following equality :

$$W_C(X+Y, X-Y) = \sum_{b \in F^n} \sum_{a \in C} (-1)^{\langle a, b \rangle} X^{n-w(b)} Y^{w(b)} \text{ (Mac Williams cf [12])}.$$

Its distance enumerator,  $D_C(X, Y) = \frac{1}{|C|} \sum_{a_1, a_2 \in C} X^{n-w(a_1+a_2)} Y^{w(a_1+a_2)}$ , satisfies :

$$D_C(X+Y, X-Y) = \frac{1}{|C|} \sum_{b \in F^n} \left( \sum_{a \in C} (-1)^{\langle a, b \rangle} \right)^2 X^{n-w(b)} Y^{w(b)}.$$

$C$  is linear if and only if for any element  $b$  of  $F^n$ , the sum  $\sum_{a \in C} (-1)^{\langle a, b \rangle}$  is equal either

to 0, or to the cardinal of  $C$  :

the condition is necessary since the functions  $a \rightarrow \langle a, b \rangle$  are then linear forms on  $C$ , and it is sufficient since, by the inverse formula of the Fourier transform, we have,

denoting by  $C^\perp$  the linear space  $\{b \in F^n / \forall a \in C, \langle a, b \rangle = 0\}$ , and by  $1_C$  the characteristic function of  $C$  :

$$\left( \forall b \in F^n, \sum_{a \in F^n} 1_C(a) (-1)^{\langle a, b \rangle} = |C| 1_{C^\perp}(b) \right) \Rightarrow$$

$$\left( \forall a \in F^n, 1_C(a) = \frac{|C|}{2^n} \sum_{b \in F^n} 1_{C^\perp}(b) (-1)^{\langle a, b \rangle} = \frac{|C| |C^\perp|}{2^n} 1_{C^{\perp\perp}}(a) \right) \Rightarrow (C = C^{\perp\perp}).$$

We have then :

$$W_C = D_C, W_{C^\perp} = D_{C^\perp}, W_C(X+Y, X-Y) = |C| W_{C^\perp}(X, Y) \text{ (Mac Williams identity)}$$

$$\text{and : } D_C(X+Y, X-Y) = |C| D_{C^\perp}(X, Y).$$

There are examples of non linear codes  $C$  and  $C^\circ$  such that the relation :

$$D_C(X+Y, X-Y) = |C| D_{C^\circ}(X, Y)$$

is still valid (cf [9], [12] ch 15).

We will say then that the codes  $C$  and  $C^\circ$  are formally dual.

The most famous is the case of the Kerdock codes and the Preparata codes of same length : Semakov and Zinov'ev proved their formal duality in 1969. But, as Paul Camion and J.H. Van Lint notice it in [2] and [15], their proof does not

explain completely why  $\frac{1}{|C|} D_C(X+Y, X-Y)$  belongs to  $\mathbb{N}[X, Y]$  when  $C$  is a Kerdock code (respectively a Preparata code), and, *a fortiori*, why it is the distance enumerator of the Preparata code (respectively the Kerdock code) of same length.

J. M. Goethals [9] gives another example of nonlinear formally dual codes : the Delsarte-Goethals codes  $\mathcal{DG}(m, (m-2)/2)$  and the Goethals codes  $\mathcal{T}(m)$ . His result, more difficult than the previous one, is proved by the same general way.

Paul Camion [2] asks for a new analytic proof of Zinov'ev and Semakov's result, and wonders if there exists a new kind of duality between (nonlinear) codes which would explain the phenomenon.

W. Kantor [10] proves that the generalized Preparata codes defined in [1] are inequivalent and concludes that the formal duality between the Kerdock codes and the Preparata codes is "merely a coincidence".

We introduce a new class of binary codes : a code  $C$  is called pseudo-linear if there exists a linear binary code  $C'$  of same length, a set of permutations  $\tau_\alpha$  on  $\{1, \dots, n\}$  ( $\alpha \in A$ ), and a permutation  $\phi$  on  $F^n$  such that, for any  $b \in F^n$ , denoting by  $b_\alpha$  the element  $(b_{\tau_\alpha(1)}, \dots, b_{\tau_\alpha(n)})$  :

- 1)  $\phi^{-1}(b_\alpha)$  has same weight as  $\phi^{-1}(b)$
- 2) 
$$\sum_{a \in C, \alpha \in A} (-1)^{\langle a_\alpha, b \rangle} = \sum_{a' \in C', \alpha \in A} (-1)^{\langle a'_\alpha, \phi(b) \rangle}.$$

We prove that the weight enumerators of the codes  $C$  and  $\phi^{-1}(C'^\perp)$  satisfy then the Mac Williams identity.

We obtain a new analytic proof of the result of Semakov and Zinov'ev (respectively Goethals) by proving that there exists codes which :

- have same weight enumerators as the Kerdock codes (respectively the  $\mathcal{DG}(m, (m-2)/2)$  codes),
- are pseudo-linear,
- admit the generalized Preparata codes of same lengths (respectively the  $\mathcal{T}(m)$  codes) as formally dual codes.

## I PSEUDO-LINEAR CODES

### DEFINITION 1

Two binary codes of same length  $C$  and  $C^\circ$  are called formally dual if their distance enumerators  $D_C$  and  $D_{C^\circ}$  satisfy the following relation :

$$D_C(X+Y, X-Y) = |C| D_{C^\circ}(X, Y).$$

### Remark

- 1) if  $n$  is the length of  $C$  and  $C^\circ$ , then  $|C^\circ| = \frac{2^n}{|C|}$  (apply the equality with  $X=Y=1$ )
- 2) the previous definition is symmetric : apply the equality where  $X$  and  $Y$  are respectively replaced by  $X+Y$  and  $X-Y$ .

### DEFINITION 2

A binary code  $C$  is called pseudo-linear if there exists a binary linear code  $C'$  of same length  $n$  and same cardinal, a set of permutations  $\tau_\alpha$  on  $\{1, \dots, n\}$  (where  $\alpha$  ranges over a set  $A$ ), and a permutation  $\phi$  on  $F^n$  such that, for any  $b = (b_1, \dots, b_n) \in F^n$ , denoting by  $b_\alpha$  the element  $(b_{\tau_\alpha(1)}, \dots, b_{\tau_\alpha(n)})$  :

1)  $\phi^{-1}(b_\alpha)$  has same weight as  $\phi^{-1}(b)$

$$2) \sum_{a \in C, \alpha \in A} (-1)^{\langle a_\alpha, b \rangle} = \sum_{a' \in C', \alpha \in A} (-1)^{\langle a', \phi(b) \rangle}.$$

$\phi$  is then called an associated permutation, and  $C'$  an associated linear code.

### Remark

Linear codes are among pseudo-linear codes those for which  $A$  may contain only one element since a binary code  $C$  is linear if and only if for any  $b \in F^n$ , the sum

$$\sum_{a \in C} (-1)^{\langle a, b \rangle} \text{ is equal either to the cardinal of } C \text{ or to } 0.$$

### THEOREM

Let  $C$  be a pseudo-linear code,  $\phi$  an associated permutation, and  $C'$  an associated linear code for  $C$ , then the weight enumerators of  $C$  and  $\phi^{-1}(C'^\perp)$  satisfy the Mac Williams identity :  $W_C(X+Y, X-Y) = |C| W_{\phi^{-1}(C'^\perp)}(X, Y)$ .

*Proof:*

For any  $\alpha$ ,  $\tau_\alpha$  is a permutation on  $\{1, \dots, n\}$ , so the code  $C_\alpha = \{a_\alpha, a \in C\}$  has same weight distribution as  $C$ , and we have :

$$W_C(X, Y) = \frac{1}{|A|} \sum_{\alpha \in A} W_{C_\alpha}(X, Y) \text{ and therefore :}$$

$$W_C(X+Y, X-Y) = \frac{1}{|A|} \sum_{a \in C, \alpha \in A} \sum_{b \in F^n} (-1)^{\langle a_\alpha, b \rangle} X^{n-w(b)} Y^{w(b)}.$$

Using property 2 of  $\phi$ , we obtain :

$$\begin{aligned}
W_C(X+Y, X-Y) &= \frac{1}{|A|} \sum_{a' \in C', \alpha \in A} \sum_{b \in F^n} (-1)^{\langle a', \phi(b) \rangle} X^{n-w(b)} Y^{w(b)} \\
&= \frac{1}{|A|} \sum_{a' \in C', \alpha \in A} \sum_{b \in F^n} (-1)^{\langle a', b \rangle} X^{n-w(\phi^{-1}(b))} Y^{w(\phi^{-1}(b))} \\
&= \frac{1}{|A|} \sum_{a' \in C', \alpha \in A} \sum_{b \in F^n} (-1)^{\langle a', b_\alpha \rangle} X^{n-w(\phi^{-1}(b_\alpha))} Y^{w(\phi^{-1}(b_\alpha))} \\
&\quad (\text{ since the mapping } b \rightarrow b_\alpha \text{ is a permutation on } F^n) \\
&= \frac{1}{|A|} \sum_{a' \in C', \alpha \in A} \sum_{b \in F^n} (-1)^{\langle a', b_\alpha \rangle} X^{n-w(\phi^{-1}(b))} Y^{w(\phi^{-1}(b))} \\
&\quad (\text{ since } w(\phi^{-1}(b_\alpha)) = w(\phi^{-1}(b))) \\
&= \sum_{a' \in C'} \sum_{b \in F^n} (-1)^{\langle a', b \rangle} X^{n-w(\phi^{-1}(b))} Y^{w(\phi^{-1}(b))} \\
&= |C| \cdot W_{\phi^{-1}(C)}(X, Y) \quad (\text{ since } \sum_{a' \in C'} (-1)^{\langle a', b \rangle} \text{ is equal to :} \\
&\quad |C'| \text{ if } b \in C'^\perp \text{ and 0 otherwise).}
\end{aligned}$$

DEFINITION 3

$\phi^{-1}(C'^\perp)$  is called *pseudo-dual* to  $C$ .

Remark

if  $\phi$  and  $C'^\perp$  satisfy the following property :

$$\forall b \in C'^\perp, \exists \alpha \in A / \forall b' \in C'^\perp, (\phi^{-1}(b) + \phi^{-1}(b'))_\alpha \in \phi^{-1}(C'^\perp),$$

then  $\phi^{-1}(C'^\perp)$  is distance invariant . Its weight and distance enumerators are then equal .

## II KERDOCK AND PREPARATA CODES

From now on, we consider codes which are sets of boolean functions on a Galois field  $G = GF(2^m)$ . This corresponds to the point of view on codes that we had in the previous paragraph since if we write  $G = \{x_1, \dots, x_n\}$  (with  $n = 2^m$ ), a boolean function  $f$  on  $G$



can be identified with the element  $a = (f(x_1), \dots, f(x_n))$  of  $F^n$ .

If a word  $a$  corresponds to a function  $f$ , then the word  $a_\alpha$  will correspond to the function  $f \circ \tau_\alpha$  where  $\tau_\alpha$  is now considered a permutation on  $G$ .

Let us recall that for  $m$  even  $\geq 4$ , the Kerdock code  $K_m$  of length  $2^m$  is the union of the Reed-Muller code of order 1,  $R(1, m)$ , with  $(2^{m-1}-1)$  cosets of  $R(1, m)$ , all constituted of bent functions belonging to  $R(2, m)$  (a function  $f$  of  $R(2, m)$  is bent if and only if its associated symplectic form :

$$(u, v) \in G^2 \rightarrow f(0) + f(u) + f(v) + f(u+v) \text{ is non-degenerated.}$$

Moreover,  $K_m$  is distance invariant.

In the following, the Galois field  $G = GF(2^m)$  is identified with the cartesian product :  $GF(2^{m'}) \times F = G' \times F$  where  $m' = m-1$ , so that  $R(1, m)$  and  $R(2, m)$  are considered as sets of boolean functions on  $G' \times F$ .

The dot product between two such functions is :

$$\langle f, g \rangle = \sum_{x \in G'} (f(x, 0) g(x, 0) + f(x, 1) g(x, 1)).$$

Let  $j$  be an integer prime to  $m'$ .

The generalized Preparata code  $\mathcal{P}_{m,j}$  is the set of all functions  $g$  on  $G' \times F$  such that (cf [1]) :

(i) the supports of the functions on  $G' : x \rightarrow g(x, 0)$  and  $x \rightarrow g(x, 1)$  have

$$\text{even cardinals, and : } \sum_{x \in G' / g(x, 0) = 1} x = \sum_{x \in G' / g(x, 1) = 1} x$$

$$(ii) \quad \sum_{x \in G' / g(x, 0) + g(x, 1) = 1} x^{2^j + 1} = \left( \sum_{x \in G' / g(x, 1) = 1} x \right)^{2^j + 1}.$$

$\mathcal{P}_{m,j}$  is distance invariant.

$\text{tr}$  denotes the trace function from  $GF(2^{m'})$  to  $F$ .

### Lemma 1

Let  $j$  be a positive integer prime to  $m'$ .

Let  $E_j$  be the set of all functions  $f_u$  defined on  $G = GF(2^{m'}) \times F$  by :

$$f_u(x, \varepsilon) = \text{tr}[(ux)^{2^j + 1} + \varepsilon ux], \text{ where } u \text{ ranges over } G'.$$

Then the code  $C_j = E_j + R(1, m)$  has same weight distribution as  $K_m$ .

*Proof :*

$f_0$  is the zero function, so  $f_0 + R(1, m) = R(1, m)$ .

For  $u \neq 0$ ,  $f_u$  belongs to  $R(2, m)$  and its associated symplectic form (cf [12] ch 15) :

$$((x, \varepsilon), (y, \eta)) \rightarrow \text{tr}[ux(uy)^{2^j} + uy(ux)^{2^j} + \varepsilon uy + \eta ux] = u[(ux)^{2^{m'-j}} + (ux)^{2^j} + \varepsilon]uy + \eta ux$$

is non-degenerated since :

1) its kernel is the linear space :

$$\begin{aligned} & \{(x, \epsilon) \in \text{GF}(2^{m'})_{\text{XF}} / (ux)^{2^{m'-j}} + (ux)^{2^j} + \epsilon = 0 \text{ and } \text{tr}(ux) = 0\} \\ &= \{(x, \epsilon) \in \text{GF}(2^{m'})_{\text{XF}} / (ux)^{2^{m'-j}} + (ux)^{2^j} = \epsilon = 0 \text{ and } \text{tr}(ux) = 0\} \text{ (since } \text{tr}((ux)^{2^{m'-j}} + (ux)^{2^j}) = 0) \\ &= \{(x, \epsilon) \in \text{GF}(2^{m'})_{\text{XF}} / (ux) + (ux)^{2^{2^j}} = \epsilon = 0 \text{ and } \text{tr}(ux) = 0\} \\ &= \{(x, \epsilon) \in \text{GF}(2^{m'})_{\text{XF}} / (ux) = 0 \text{ or } (ux)^{2^{2^j-1}} = 1, \epsilon = 0 \text{ and } \text{tr}(ux) = 0\} ; \end{aligned}$$

2)  $2^j$  being prime to  $m'$ ,  $2^{2^j-1}$  is prime to  $2^{m'-1}$ , and the mapping  $x \rightarrow x^{2^{2^j-1}}$  is one to one. So, the kernel is equal to :

$$\{(x, \epsilon) \in \text{GF}(2^{m'})_{\text{XF}} / ux = 0 \text{ or } 1, \epsilon = 0 \text{ and } \text{tr}(ux) = 0\} = \{(0, 0)\}.$$

Therefore, if  $u \neq 0$ , all the elements of  $f_u + R(1, m)$  are bent functions, and  $C$  has same weight distribution as  $K_m$ .

#### PROPOSITION 1

*For any  $j$  prime to  $m'$ , the code  $C_j$  is pseudo-linear and  $\mathcal{P}_{m,j}$  is pseudo-dual to  $C_j$ .*

*Proof :*

Let  $E'_j$  be the set of all the functions  $f'_u$  defined on  $G = \text{GF}(2^{m'})_{\text{XF}}$  by :

$$f'_u(x, \epsilon) = \text{tr}[(ux)^{2^j+1}], \text{ where } u \text{ ranges over } G'.$$

Let  $C'_j$  be the code equal to :  $E'_j + R(1, m)$ .

$u^{2^j+1}$  ranges over all  $G'$  when  $u$  ranges over  $G'$ , therefore  $E'_j$  is the

set of all functions of the type :  $(x, \epsilon) \rightarrow \text{tr}[vx^{2^j+1}]$ , ( $v \in G'$ ), and  $C'_j$  is a linear code .

Let  $\tau_\alpha$ , ( $\alpha \in G'$ ), be the permutation on  $G$  :  $(x, \epsilon) \rightarrow (x + \epsilon \alpha, \epsilon)$  equal to  $(x, 0)$  if  $\epsilon = 0$ , and  $(x + \alpha, 1)$  if  $\epsilon = 1$ .

Let us recall (cf [12], [5]) that  $R(1, m)^\perp = R(m-2, m)$  is the set of all boolean functions on  $G'_{\text{XF}}$  such that the supports of the functions on  $G' : x \rightarrow g(x, 0)$  and  $x \rightarrow g(x, 1)$  have

$$\text{even cardinals, and that : } \sum_{x \in G'/g(x,0)=1} x = \sum_{x \in G'/g(x,1)=1} x .$$

If a boolean function  $g$  on  $G$  does not belong to  $R(1, m)^\perp$ , we have (cf [9]) :

$$\sum_{f \in C, \alpha \in G'} (-1)^{\langle f \circ \tau_\alpha, g \rangle} = \sum_{f' \in C', \alpha \in G'} (-1)^{\langle f' \circ \tau_\alpha, g \rangle} = 0.$$

If  $g$  belongs to  $R(1, m)^\perp$ , then :

$$\begin{aligned} \sum_{f \in C, \alpha \in G'} (-1)^{\langle f \circ \tau_\alpha, g \rangle} &= \sum_{u \in G', \alpha \in G'} \sum_{h \in R(1, m)} (-1)^{\langle f_u \circ \tau_\alpha + h, g \rangle} \\ &= |R(1, m)| \sum_{u \in G', \alpha \in G'} (-1)^{\langle f_u \circ \tau_\alpha, g \rangle} \end{aligned}$$

since we know that  $R(1, m) \circ \tau_\alpha = R(1, m)$  and that  $R(1, m)$  is linear.

### Lemma 2

Let  $\{f_u, u \in A\}$  be a set of elements of  $R(2, m)$ ,  $h_u$  and  $k_u$  the functions on  $G'$  such that:  $f_u(x, \varepsilon) = h_u(x) + \varepsilon k_u(x)$  ( $h_u \in R(2, m')$ ,  $k_u \in R(1, m')$ ), and  $F_u$  the kernel of the symplectic form  $\varphi_u$  associated with  $h_u$ . Let  $\tau_\alpha$  be the permutation on  $G : (x, \varepsilon) \rightarrow (x + \varepsilon\alpha, \varepsilon)$ .

Then, for any element  $g$  of  $R(1, m)^\perp$ , if we denote by  $\lambda_g$  the sum  $\sum_{x \in G' / g(x, 1) = 1} x$ ,

we have :

$$\sum_{u \in A, \alpha \in G'} (-1)^{\langle f_u \circ \tau_\alpha, g \rangle} = |G'| \sum_{u \in A / \lambda_g \in F_u} (-1)^{\langle f_u, g \rangle}.$$

*Proof :*

Let  $g_0(x) = g(x, 0)$  and  $g_1(x) = g(x, 1)$ .

We have, denoting by  $t_\alpha$  the translation on  $G'$  of vector  $\alpha$  :

$$\begin{aligned}
\sum_{u \in A, \alpha \in G'} (-1)^{\langle f_u \circ \tau_\alpha, g \rangle} &= \sum_{u \in A, \alpha \in G'} (-1)^{\langle h_u, g_0 \rangle + \langle h_u \circ t_\alpha, g_1 \rangle + \langle k_u, g_1 \rangle} \\
\sum_{u \in A, \alpha \in G'} (-1)^{\langle h_u, g_0 + g_1 \rangle + \varphi_u(\alpha, \lambda_g) + \langle k_u, g_1 \rangle} &= |G'| \sum_{u \in A / \lambda_g \in F_u} (-1)^{\langle f_u, g \rangle} \\
(\text{since } \varphi_u(\alpha, \lambda_g) &= \sum_{x \in /g_1(x)=1} \varphi_u(\alpha, x) = \sum_{x \in /g_1(x)=1} \left( h_u(0) + h_u(\alpha) + h_u(x) + h_u(x+\alpha) \right) = \\
&\langle h_u + h_u \circ t_\alpha, g_1 \rangle \text{ and since the linear form } \alpha \rightarrow \varphi_u(\alpha, \lambda_g) \text{ is either null or balanced on } G')
\end{aligned}$$

⌊

Here we have :

$$h_u(x) = \text{tr} \left( (ux)^{2^j+1} \right), \varphi_u(x, y) = \text{tr} \left( u^{2^j+1} (x^{2^j} y + y^{2^j} x) \right) = \text{tr} \left( uy \left( (ux)^{2^j} + (ux)^{2^{m-j}} \right) \right)$$

$$F_u = \left\{ 0, \frac{1}{u} \right\} \text{ if } u \neq 0, \text{ and } G' \text{ if } u = 0.$$

Therefore :

$$\begin{aligned}
\sum_{f \in C_j, \alpha \in G'} (-1)^{\langle f \circ \tau_\alpha, g \rangle} &= \\
|R(1, m)| |G'| \sum_{u \in G' / \lambda_g u = 0 \text{ or } 1} (-1)^{\text{tr} \left( \sum_{x \in G' / g(x,0)+g(x,1)=1} (ux)^{2^j+1} + \sum_{x \in G' / g(x,1)=1} (ux) \right)}.
\end{aligned}$$

For the same reasons :

$$\begin{aligned}
\sum_{f' \in C'_j, \alpha \in G'} (-1)^{\langle f' \circ \tau_\alpha, g \rangle} &= \\
|R(1, m)| |G'| \sum_{u \in G' / \lambda_g u = 0 \text{ or } 1} (-1)^{\text{tr} \left( \sum_{x \in G' / g(x,0)+g(x,1)=1} (ux)^{2^j+1} \right)}.
\end{aligned}$$

Let  $k$  be any element of  $R(1, m)^\perp$  such that :

$$\sum_{x \in G' / k(x,0)=1} x^{2^j+1} = 1 \text{ and } \forall x \in G', k(x,1)=0.$$

For any element  $\lambda$  of  $G'$ , different from 0, let  $k_\lambda$  be defined by :  $k_\lambda(x, \varepsilon) = k(x/\lambda, \varepsilon)$ .

We have :

$$\sum_{x \in G' / k_\lambda(x, 0) = 1} x^{2^j+1} = \lambda^{2^j+1} \text{ and } \forall x \in G', k_\lambda(x, 1) = 0.$$

We can take :  $\phi(g) = g$  if  $\lambda_g = 0$  or if  $g \notin R(1, m)^\perp$ , and  $\phi(g) = g + k_{\lambda_g}$  otherwise since the relation :

$$\sum_{f \in C_j, \alpha \in G'} (-1)^{\langle f \circ \tau_\alpha, g \rangle} = \sum_{f' \in C'_j, \alpha \in G'} (-1)^{\langle f' \circ \tau_\alpha, \phi(g) \rangle}$$

stands : if  $g \in R(1, m)^\perp$  and  $\lambda_g \neq 0$ ,

$$\lambda_g u = 1 \Rightarrow \sum_{x \in G'/g(x, 1) = 1} ux = 1 = \sum_{x \in G'/k_{\lambda_g}(x, 0) + k_{\lambda_g}(x, 1) = 1} (ux)^{2^j+1}.$$

So, condition 2 of the definition is satisfied by  $\phi$ .

$\phi$  is a permutation :  $\phi^{-1} = \phi$ .

Condition 1 is clearly fulfilled since :  $(\forall x \in G', k(x, 0) = 0) \Rightarrow (\phi(g \circ \tau_\alpha) = \phi(g) \circ \tau_\alpha)$ .

So, all the conditions of the definition are fulfilled.

Let us prove now that  $\phi^{-1}(C'_j) = \phi(C'_j)$  is equal to the generalized Preparata code  $\mathcal{P}_{m,j}$ .  
Condition (i) of the definition of  $\mathcal{P}_{m,j}$  is equivalent to :  $g \in R(m-2, m)$ .

$C'_j$  is the set of all the functions  $g \in R(m-2, m)$  such that :  $\forall u \in G', \langle f'_u, g \rangle = 0$ ,  
or, equivalently :

$$\forall u \in G', \text{tr} \left( \sum_{x \in G'/g(x, 0) + g(x, 1) = 1} (ux)^{2^j+1} \right) = 0,$$

$$\forall v \in G', \text{tr} \left( v \sum_{x \in G'/g(x, 0) + g(x, 1) = 1} x^{2^j+1} \right) = 0,$$

$$\sum_{x \in G'/g(x, 0) + g(x, 1) = 1} x^{2^j+1} = 0.$$

For any  $g \in R(m-2, m)$  such that  $\lambda_g = 0$ ,  $g$  belongs to  $C'_j$  if and only if  $g$  belongs to  $\mathcal{P}_{m,j}$ .

Any  $g \in R(m-2, m)$  such that  $\lambda_g \neq 0$  belongs to  $C'_j$  if and only if  $\phi(g)$  belongs to  $\mathcal{P}_{m,j}$  since :

$$\sum_{x \in G'/\phi(g)(x,0)+\phi(g)(x,1)=1} x^{2^j+1} = \sum_{x \in G'/g(x,0)+g(x,1)=1} x^{2^j+1} + \lambda_g^{2^j+1}.$$

This finishes the proof.

Notice that  $C'_j^\perp$  and  $\phi$  satisfy the condition stated in the remark which follows the theorem :

If  $g$  and  $g'$  are two elements of  $C'_j^\perp$ , then we have :

$$\phi(\phi^{-1}(g) + \phi^{-1}(g')) \circ \tau_{\lambda_g} = \phi(\phi(g) + \phi(g')) \circ \tau_{\lambda_g} =$$

$$g + g \circ \tau_{\lambda_g} + g' + g' \circ \tau_{\lambda_g} + k_{\lambda_g} + k_{\lambda_{g'}} + k_{\lambda_g + \lambda_{g'}}.$$

and it is easy to check that this function belongs to  $C'_j^\perp$ .

Since the Kerdock and the Preparata codes are both distance invariant, we deduce :

**COROLLARY 1**

*The Kerdock codes and the generalized Preparata codes of same length are formally dual.*

### III DELSARTE-GOETHALS AND GOETHALS CODES

Another example of nonlinear formally dual codes (cf [8], [9]) is the case of Delsarte-Goethals

**DG**(m,(m-2)/2) and Goethals **T**(m) codes :

$m = 2t + 2 \geq 4$ , **T**(m) is the intersection of  $\mathcal{P}_{m,t}$  and  $\mathcal{P}_{m,t-1}$  and **DG**(m,t) is the set of all boolean functions on  $G$  of the type :

$$(x, \varepsilon) \rightarrow \text{tr} \left( \sum_{j=1}^t (ux)^{2^j+1} + vx^3 + \varepsilon ux \right) + l(x, \varepsilon)$$

where  $u$  and  $v$  range over  $G'$ , and  $l$  over  $R(1, m)$ . The symplectic form associated with any of these functions (with  $u \neq 0$  or  $v \neq 0$ ) has a rank at least equal to  $2t$ .

Moreover, the symplectic form associated with the sum of two different functions of this type

has a rank at least equal to  $2t$ , and **DG**(m,t) has as many elements as any code having this property (ie it is optimal, just as the Kerdock code). The distance distribution of such a code is

fixed by this property (cf [8]).

Both codes are distance invariant.

#### Lemma 3

For any  $u$  and  $v$  in  $G'$ , let  $f_{u,v}$  be the function on  $G$  defined by :

$$\forall (x, \varepsilon) \in G, f_{u,v}(x, \varepsilon) = \text{tr} (ux^{2^{t+1}} + vx^{2^{t-1}+1}) + \varepsilon h_{u,v}(x)$$

where  $h_{u,v}$  is any linear form on  $G'$  whose restriction to the kernel of the symplectic form

$\phi_{u,v}$  associated with the function  $f_{u,v} : (x, \varepsilon) \rightarrow \text{tr} (ux^{2^{t+1}} + vx^{2^{t-1}+1})$  takes the same

values as  $f'_{u,v}$  itself .

Then the code  $D = \{ f_{u,v} \mid u,v \in G' \} + R(1,m)$  has same weight distribution as  $DG(m,t)$  .

*Proof :*

Notice that there exists such an  $h_{u,v}$  since we know that the restriction of a quadratic function to the Kernel of its associated symplectic form is affine (here, linear).

We have :

$$\begin{aligned} \phi_{u,v}(x,y) &= \text{tr} [u (x^{2^t}y + y^{2^t}x) + v (x^{2^{t-1}}y + y^{2^{t-1}}x)] \\ &= \text{tr} [(u x^{2^t} + (ux)^{2^{m'-t}} + v x^{2^{t-1}} + (vx)^{2^{m'-t+1}})y] \\ &= \text{tr} [(u x^{2^t} + (ux)^{2^{t+1}} + v x^{2^{t-1}} + (vx)^{2^{t+2}})y] \end{aligned}$$

and the kernel of  $\phi_{u,v}$  is the linear space :

$$\begin{aligned} \{x \in G' / u x^{2^t} + (ux)^{2^{t+1}} + v x^{2^{t-1}} + (vx)^{2^{t+2}} = 0\} = \\ \{x \in G' / u^{2^{t+2}} x^2 + (ux)^{2^2} + v^{2^{t+2}} x + (vx)^{2^3} = 0\} = \\ \{x \in G' / (vx)^8 + (ux)^4 + u^{2^{t+2}} x^2 + v^{2^{t+2}} x = 0\} . \end{aligned}$$

Therefore, the dimension of this kernel is at most 3 and the rank of  $\phi_{u,v}$  is at least equal to  $m'-3 = m-4$ .

Let  $\psi_{u,v}$  the symplectic form associated with  $f_{u,v}$  .

We know (cf Delsarte-Goethals [8])) that  $\psi_{u,v}$  's rank is equal to  $\phi_{u,v}$  's one if  $h_{u,v}$  takes always the 0 value on the kernel of  $\phi_{u,v}$  , or, equivalently (cf [5]) if the function  $f_{u,v}$  is not balanced on  $G'$ . Otherwise  $\psi_{u,v}$  's rank is equal to  $\phi_{u,v}$  's one plus 2.

Let a, b, c, and d be the cardinals respectively of the sets :

$$\begin{aligned} \{(u,v) \in G'^2 / \text{rank } \psi_{u,v} = m, \text{rank } \phi_{u,v} = m-2\} &= \{(u,v) \in G'^2 / f_{u,v} \text{ balanced, rank } \phi_{u,v} = m-2\} \\ \{(u,v) \in G'^2 / \text{rank } \psi_{u,v} = m-2, \text{rank } \phi_{u,v} = m-2\} &= \{(u,v) \in G'^2 / f_{u,v} \text{ not balanced, rank } \phi_{u,v} = m-2\} \\ \{(u,v) \in G'^2 / \text{rank } \psi_{u,v} = m-2, \text{rank } \phi_{u,v} = m-4\} &= \{(u,v) \in G'^2 / f_{u,v} \text{ balanced, rank } \phi_{u,v} = m-4\} \\ \{(u,v) \in G'^2 / \text{rank } \psi_{u,v} = m-4, \text{rank } \phi_{u,v} = m-4\} &= \{(u,v) \in G'^2 / f_{u,v} \text{ not balanced, rank } \phi_{u,v} = m-4\} . \end{aligned}$$

According to [9], table IV, d is equal to 0.

We want to prove that the rank distribution of the set  $\{\psi_{u,v}, u,v \in G'\}$  is the same as the rank distribution of the set of the symplectic forms  $\rho_{u,v}$  associated with the functions

$$(x, \varepsilon) \rightarrow \text{tr} \left( \sum_{j=1}^t (ux)^{2^{j+1}} + vx^3 + \varepsilon ux \right).$$

Let us recall that for any symplectic form  $\varphi$  on  $G'$  of rank  $2h$ , we have (cf [5]):

$$\left( \sum_{x, y \in G'} (-1)^{\varphi(x,y)} \right) = 2^{2m'-2h}$$

and that if  $f'$  is a quadratic function on  $G'$ , of symplectic form  $\varphi$ :

$$\left( \sum_{x \in G'} (-1)^{f'(x)} \right)^2 = \left( \sum_{x, y \in G'} (-1)^{\varphi(x,y)} \right)^2 \text{ if } f' \text{ is not balanced, and } 0 \text{ otherwise.}$$

So,  $a$ ,  $b$  and  $c$  satisfy the following relations :

$$1) \ a + b + c + 1 = 2^{2m-2}$$

$$\begin{aligned} 2) \ (a+b) 2^m + c 2^{m+2} + 2^{2m-2} &= \sum_{u,v \in G'} \left( \sum_{x,y \in G'} (-1)^{\varphi_{uv}(x,y)} \right) = \\ &= \sum_{u,v \in G'} \left( \sum_{x,y \in G'} (-1)^{\text{tr} (u[x^{2^t}y+y^{2^t}x] + v[x^{2^{t-1}}y+y^{2^{t-1}}x])} \right) = \\ &= |G'|^2 \left| \{x,y \in G' / x^{2^t}y+y^{2^t}x = 0 \text{ and } x^{2^{t-1}}y+y^{2^{t-1}}x = 0\} \right| = |G'|^2 \left| \{x,y \in G' / x = 0 \text{ or } y = 0 \text{ or } x = y\} \right| = \\ &= |G'|^2 (3 |G'| - 2) = 2^{2m-2} (3 \times 2^{m-1} - 2). \end{aligned}$$

$$\begin{aligned} 3) \ b 2^m + 2^{2m-2} &= \sum_{u,v \in G'} \left( \sum_{x,y \in G'} (-1)^{\text{tr} (ux^{2^{t+1}} + vx^{2^{t-1}+1})} \right)^2 = \\ &= \sum_{u,v \in G'} \left( \sum_{x,y \in G} (-1)^{\text{tr} (u[x^{2^{t+1}}+y^{2^{t+1}}] + v[x^{2^{t-1}+1}+y^{2^{t-1}+1}])} \right) = \\ &= |G'|^2 \left| \{x,y \in G' / x^{2^{t+1}} = y^{2^{t+1}} \text{ and } x^{2^{t-1}+1} = y^{2^{t-1}+1}\} \right| = \\ &= |G'|^3 = 2^{3m-3}. \end{aligned}$$

From these equalities, we deduce :

$$a = (2^{m-1}-1)(2^{m-1}+4) / 3, \text{ and } b+c = (2^{m-1}-1)(2^{m-1}) / 3,$$



and therefore, according to [12], ch 15, th 5, the weight distribution of  $D$  is :

$$\begin{array}{ll}
 i = & \text{number of the elements of weight } i = \\
 0 \text{ or } 2^{2t+2} & 1 \\
 2^{2t+1} \pm 2^{t+1} & 2^{2t}(2^{2t+1}-1)(2^{2t+2}-1) / 3 \\
 2^{2t+1} \pm 2^t & 2^{2t+2}(2^{2t+1}-1)(2^{2t+1}+4) / 3 \\
 2^{2t+1} & 2(2^{2t+2}-1)(2^{4t+1}-2^{2t+1})
 \end{array}$$

which is the same as  $\mathbf{D}\mathbf{G}(m,t)$ 's one, according to Goethals ([9], table III).

## PROPOSITION 2

*The code  $D$  is pseudo-linear and  $\mathcal{T}(m)$  is pseudo-dual to  $D$ .*

*Proof :*

Let  $E'$  be the set of all the functions  $f'_{u,v}$ .

Let  $D'$  be the linear code equal to :  $E' + R(1, m)$ .

$\tau_\alpha$  ( $\alpha \in G'$ ) is as before the permutation on  $G : (x, \varepsilon) \rightarrow (x + \varepsilon \alpha, \varepsilon)$ .

Let us define  $\phi$  :

If a function  $g$  does not belong to  $R(1, m)^\perp$ , then :  $\phi(g) = g$ .

If  $g$  belongs to  $R(1, m)^\perp$ , then :

$$\begin{aligned}
 \sum_{f \in D, \alpha \in G'} (-1)^{\langle f \circ \tau_\alpha, g \rangle} &= \sum_{u,v \in G', \alpha \in G'} \sum_{h \in R(1, m)} (-1)^{\langle f_{u,v} \circ \tau_\alpha + h, g \rangle} \\
 &= |R(1, m)| \sum_{u,v \in G', \alpha \in G'} (-1)^{\langle f_{u,v} \circ \tau_\alpha, g \rangle}
 \end{aligned}$$

Let  $F_{u,v}$  be the kernel of  $\phi_{u,v}$ . By Lemma 2, we have :

$$\begin{aligned}
 \sum_{u,v \in G', \alpha \in G'} (-1)^{\langle f_{u,v} \circ \tau_\alpha, g \rangle} &= |G'| \sum_{u,v \in G'/\lambda_g \in F_{u,v}} (-1)^{\langle f_{u,v}, g \rangle} \\
 &= |G'| \sum_{u,v \in G'/\lambda_g \in E_{u,v}} (-1)^{\langle f_{u,v}, g \circ \sigma + h_{u,v}(\lambda_g) \rangle}, \text{ since } h_{u,v} \text{ is linear} \\
 &= |G'| \sum_{u,v \in G'/\lambda_g \in E_{u,v}} (-1)^{\langle f_{u,v}, g \circ \sigma + f_{u,v}(\lambda_g) \rangle}.
 \end{aligned}$$

Let  $k$  be any element of  $R(1, m)^\perp$  such that :

$$\sum_{x \in G' / k(x,0) = 1} x^{2^{i+1}} = \sum_{x \in G' / k(x,0) = 1} x^{2^{i-1}+1} = 1 \text{ and } \forall x \in G', k(x,1)=0.$$

For any element  $\lambda$  of  $G'$ , different from 0, let  $k_\lambda$  be defined by :  $k_\lambda(x, \epsilon) = k(x/\lambda, \epsilon)$ .

We have :

$$\sum_{x \in G' / k_\lambda(x,0) = 1} x^{2^{i+1}} = \lambda^{2^{i+1}}, \quad \sum_{x \in G' / k_\lambda(x,0) = 1} x^{2^{i-1}+1} = \lambda^{2^{i-1}+1} \text{ and } \forall x \in G', k_\lambda(x,1)=0.$$

We have :  $f_{u,v}(\lambda) = \langle f_{u,v}, k_\lambda \rangle$ , and so, if we take :  $\phi(g) = g$  if  $\lambda_g = 0$ , and

$\phi(g) = g + k_{\lambda_g}$  otherwise, then  $D$  is clearly pseudo-linear, and  $\phi$  is an associated permutation with  $D$ .

Let us prove now that  $\phi^{-1}(D'^\perp) = \phi(D'^\perp)$  is equal to  $\mathcal{T}(m)$ .

$D'^\perp$  is the set of all the functions  $g \in R(m-2, m)$  such that :  $\forall u, v \in G', \langle f_{u,v}, g \rangle = 0$ , or, equivalently :

$$\forall u, v \in G', \text{tr} \left( \sum_{x \in G' / g(x,0)+g(x,1) = 1} (u x^{2^{i+1}} + v x^{2^{i-1}+1}) \right) = 0, \text{ and so :}$$

$$\sum_{x \in G' / g(x,0)+g(x,1) = 1} x^{2^{i+1}} = 0, \text{ and } \sum_{x \in G' / g(x,0)+g(x,1) = 1} x^{2^{i-1}+1} = 0.$$

For any  $g \in R(m-2, m)$  such that  $\lambda_g = 0$ , and any  $g \notin R(m-2, m)$ ,  $g$  belongs to  $D'^\perp$  if and only if  $g$  belongs to  $\mathcal{T}(m)$ .

Clearly, any  $g \in R(m-2, m)$  such that  $\lambda_g \neq 0$  belongs to  $D'^\perp$  if and only if  $\phi(g)$  belongs to  $\mathcal{T}(m)$ .

This finishes the proof.

Notice that  $D'^\perp$  and  $\phi$  satisfy the condition stated in the remark which follows the theorem.

**COROLLARY 2**

$\mathcal{T}(m)$  and  $\mathcal{DA}(m, t)$  are formally dual.

## CONCLUSION

We have obtained a generalization of the Mac Williams' result to some nonlinear codes . The notion of pseudo-linearity gives an explanation of the most known formal dualities between nonlinear codes .

In this new framework, we lose the nice symmetry which exists between a linear code and its dual code, since the pseudo-dual code does not seem to be always pseudo-linear.

Moreover, a pseudo-linear code does not need to be distance invariant (as we see it on the codes  $C_j$  and  $D$  met in lemmas 1 and 2), but its pseudodual code is distance invariant under a

rather natural condition stated in the remark which follows the theorem (this condition is satisfied in the cases studied in Propositions 1 and 2).

Nevertheless, this may lead to new distance invariant nonlinear codes (which are known to be

good candidates for optimum properties) whose weight enumerators and dual weight enumerators

would be both "known".

## REFERENCES

- (1) Ronald D. Baker, Jacobus H. Van Lint and Richard M. Wilson, "On the Preparata and Goethals codes" IEEE Trans Inform Theory. Vol IT 29
- (2) P. Camion, "Codes de Preparata et Codes de Kerdock" théorie des codes ENSTA, pp 21-29, 1979.
- (3) C. Carlet, "An analytic proof of the formal duality between Kerdock and Preparata Codes", Publication du L.I.T.P. n° 89-19 (L.I.T.P. 4 place Jussieu, 75251 Paris Cédex 05).
- (4) C. Carlet, "Le groupe d'automorphismes du code de Kerdock" C. R. Acad. Sci. Paris t.309, Série I, p. 843-845, 1989.
- (5) C. Carlet, thèse, "codes de Reed-Muller , codes de Kerdock et de Preparata ", Publication du L.I.T.P., 1990.
- (6) C. Carlet, "A general case of formal duality between binary nonlinear codes", Publication du L.I.T.P. n° 90-59.
- (7) P. Charpin, Thèse, "Codes cycliques étendus invariants sous le groupe affine", Rapport du L.I.T.P. n°87-6.
- (8) P. Delsarte, J.-M. Goethals, "Alternating bilinear forms over  $GF(q)$ , J. Combin. Theory, 19 A (1975) 26-50 [15, 21, A]
- (9) J.M. Goethals, "Nonlinear codes defined by quadrature forms over  $GF(2)$ ", Information and control, 31 (1976) 43-74.
- (10) W. M. Kantor, "On the inequivalence of generalized Preparata Codes", IEEE Trans Inform Theory. Vol IT.29, pp 345-348, May 1983.
- (11) A. M. Kerdock, "A class of low-rate non linear codes", Information and Control, 20 (1972), pp 182-187.

- (12) F. J. Mac Williams and N. J. Sloane, "The theory of error-correcting codes", Amsterdam, North Holland.
- (13) F. P. Preparata, "A class of optimum non linear double-error correcting codes" Information and Control, 13 (1968), pp 378-400.
- (14) J. Simonis "Reed Muller codes", Report of the faculty of math.and information n° 87-23 Delft (1987).
- (15) J. H. Van Lint, "Kerdock codes and Preparata codes", Congressus Numerantium, Vol. 39 (1983), pp 25-41.
- (16) J. H. Van Lint, "Coding Theory", Springer Verlag 201.
- (17) H.C.A. Van Tilborg "On weights in codes" Report 71-WSK03, Dep.of math. Technological University of Eindhoven, Netherlands (1971).
- (18) J. Wolfmann, "Aspects géométriques et combinatoires de l'étude des codes correcteurs"  
Thèse de Doctorat d'Etat, Université Paris VII, (1978)





ISSN 0249 - 6399